derived Weil representation

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Abstract

abstract

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1 introduction

1.1 backgrounds

The Weil representation is a special representation of symplectic group. The finite field case is defined as follows: let V be a symplectic vector space over the finite field $k = \mathbb{F}_q$ with odd characteristic, and $\mathrm{Heis}(V)$ be the Heisenberg group defined by the symplectic form:

$$1 \to k \to \mathrm{Heis}(V) \to V \to 1.$$

For a character $\psi\colon \to \mathbb{C}^\times$, we can define an irreducible representation $H_{V,\psi}$ of $\mathrm{Heis}(V)$ with central character ψ. It is a subspace of functions on the set V and its dimension is $q^{\frac{1}{2}\dim V}$. This can be extend to a projective representation ω_ψ of $\mathrm{Sp}(V)$, called the Weil representation. In general, it can be descent to a representation of the double cover $\widetilde{\mathrm{Sp}}(V)$ of the symplectic group. The case in local field $k((t))$ is defined similarly using residue.

A dual pair (G, H) is the subgroup $G \times H \to \mathrm{Sp}(V)$ such that they are the centralizer of each other. Examples are $(\mathrm{Sp}(V_1), \mathrm{O}(V_2))$ where $V = V_1 \otimes V_2$, and $(\mathrm{GL}(L_1), \mathrm{GL}(L_2))$ where $V = \mathrm{Hom}(L_1, L_2) \oplus \mathrm{Hom}(L_2, L_1)$. By restricting Weil representation to this subgroup, we obtain $\mathrm{Weil}_{G,H}$ as a representation of $G \times H$. Associated to it, we can define theta functions and construct theta lifts by using it as an integral kernel.

1.2 geometrization

By choosing a Lagrangian $L \subset V$, the Weil representation can be identified with L^2-functions on L or V/L. Thus it has a natural categorification $D(L)$. In [14], the action of $D(\mathrm{Sp}(V))$ is constructed via the functor

$$D(\mathrm{Sp}(V)) \to \mathrm{End}(D(L)) \simeq D(L \times L) \simeq D(V)$$

giving by a sheaf in $D(\mathrm{Sp}(V) \times V)$.

In the local field case, one geometric model of Weil representation is constructed in [19], but the action of Hecke categories is not written down explicitly in [19]. While in [21], the action of Sat_G and Sat_H are given separately by choosing two Lagrangians and using their Schrödinger models.

When studying Weil representations, we would expect more compatibilities such as the commutativity of these two actions. By mimicking the lattice model of the Weil representation, I could define the derived Weil category with the action of Hecke categories of $G \times H$ at the same time.

Theorem 1. *Let F be a local field and $\mathcal{O}$ its ring of integers. For a variety X, let $X_{\mathcal{O}}$ be its arc space and X_F be its loop space.*

Let $\mathrm{Weil}_{G,H}$ be the category of $G_{\mathcal{O}} \times H_{\mathcal{O}}$-equivariant $(V_{\mathcal{O}}, \psi)$-equivariant sheaves on V_F. Let $\mathrm{Sat}_G = D_{G_{\mathcal{O}}}(G_F/G_{\mathcal{O}})$ be the derived Satake category. Then we have the action of $\mathrm{Sat}_{G\times H} \simeq \mathrm{Sat}_G \otimes \mathrm{Sat}_H$ on $\mathrm{Weil}_{G,H}$. Hence the actions of Sat_G and Sat_H commute in the strongest sense.

In section 2, we will define the unramified part of the Weil representation in the derived category setting. Let $\mathrm{Weil}_{G,H}$ be the category of $G_{\mathcal{O}} \times H_{\mathcal{O}}$-equivariant Weil representation.

In [21], Lysenko constructed the functor

$$\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G^\vee) \to (\mathrm{Weil}^\heartsuit_{G,H})^{\mathrm{ss}},$$

and showed that this is an equivalence in the case of $(\mathrm{GL}_n, \mathrm{GL}_m)$-case and conjectured it is also true in the $(\mathrm{Sp}_{2m}, \mathrm{SO}_{2n})$-cases. We will show this conjecture is true in section 3.

Theorem 2. *All the irreducible objects in $\mathrm{Weil}^\heartsuit_{G,H}$ are given by $W *_G \delta_V$ for $W \in \mathrm{Irr}(G^\vee)$.*

2 definition of the categories

2.1 notations

Let k be an algebraically closed field used in the definition of geometric object. Let $\Lambda = \overline{\mathbb{Q}_\ell}$ or $\mathbb{C}$ be the field of the coefficient of sheaves. $\psi\colon k \to \Lambda^\times$ is a non-trivial character. Then we get the Artin-Schreier sheaf $\mathcal{L}_\psi \in D(\mathbb{A}^1)$. In the case $k = \Lambda = \mathbb{C}$, this is the exponential D-module.

$F = k((t))$ is the field of Laurent series, and $\mathcal{O} = k[[t]]$ is the ring of integers in F. ψ naturally extends to a character of F via residue: $\psi\colon F \xrightarrow{\mathrm{res}} k \xrightarrow{\psi} \Lambda^\times$.

When V is a symplectic vector space, use $\omega\colon V \times V \to k$ to denote the symplectic pairing. It naturally extends to a symplectic pairing on V_F:

$$V_F \times V_F \to F \xrightarrow{\mathrm{res}} k,$$

which also gives a pairing on $t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}}$. By abuse of notation, we still use ω to denote them.

For an algebraic group G, define $\mathrm{Gr}_G = G_F/G_{\mathcal{O}}$ be the affine Grassmannian of G and define $\mathrm{Sat}_G = D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ be the derived Satake category.

If needed, all categories are assumed to be $(\infty,1)$-categories. By saying derived category, we mean stable $(\infty,1)$-categories. For a derived category $\mathcal{C}$ with certain t-structure, we use $\mathcal{C}^\heartsuit$ to denote the heart of this t-structure.

2.2 Schrödinger model

In the general linear group case, the vector space V has a polarization $V = T^*L$ such that $L = \mathrm{Hom}(V_1, V_2)$ is a representation of $G \times H$. In this case, the Weil representation can be identified with $G_{\mathcal{O}} \times H_{\mathcal{O}}$-equivariant sheaves on L_F. More concretely, it is defined as a colimit of categories of the diagram:

$$\cdots \to D_{G_{2r}\times H_{2r}}(t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}}) \to D_{G_{2r+2}\times H_{2r+2}}(t^{-r-1}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}}) \to \cdots .$$

The arrows are given by $i_*p^\dagger = i_*p^*[\dim L]$, where $p\colon t^{-r}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}} \to t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}}$ is the projection and $i\colon t^{-r}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}} \to t^{-r-1}L_{\mathcal{O}}/t^{r+1}L_{\mathcal{O}}$ is the inclusion. The degree is chosen such that the middle perverse t-structure is preserved.

2.3 lattice model

When the case V is possibly not canonically split, the above construction lacks the equivariance structure. We propose another approach through the so-called lattice model. We first explain our construction through the finite case.

2.3.1 finite case

Pick any Lagrangian $L \subset V$, we can think of L as a group acting on V via addition. Then we have a relative character on L: $L \times V \xrightarrow{\psi\circ\omega} \Lambda^\times$ and corresponding sheaf $\omega^*\mathcal{L}_\psi$. Call a sheaf $\mathcal{F}$ is (L, ψ)-equivariant if we have an isomorphism

$$\mathrm{act}^*\mathcal{F} \cong \mathrm{proj}^*\mathcal{F} \otimes \omega^*\mathcal{L}_\psi.$$

Hence we can form the category $D(V/(L, \psi))$ of (L, ψ)-equivariant sheaves on V.

2.3.2 local case

Consider the $G_{\mathcal{O}} \times H_{\mathcal{O}}$-stable Lagrangian $V_{\mathcal{O}} \subset V_F$. To mimic the finite case, we want a category $D(V_F/(V_{\mathcal{O}}, \psi))$. As the colimit of finite cases, we define this category as the colimit of the following diagram:

$$\cdots \to D((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}}, \psi)) \xrightarrow{i_*p^\dagger} D((t^{-r-1}V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}})/(V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}}, \psi)) \to \cdots .$$

Even G_r can act on the space $t^{-r}V_{\mathcal{O}}/V_{\mathcal{O}}$, it cannot act on $(V_{\mathcal{O}}/t^rV_{\mathcal{O}}, \psi)$-equivariant sheaves on $t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}}$. Rather, we only have the action of G_{2r}. Hence the unramified Weil representation $D_{G_{\mathcal{O}}\times H_{\mathcal{O}}}(V_F/(V_{\mathcal{O}}, \psi))$ is the colimit of the following diagram:

$$\begin{aligned}\cdots \to D_{G_{2r}\times H_{2r}}((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}}, \psi)) \to \\ \to D_{G_{2r+2}\times H_{2r+2}}((t^{-r-1}V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}})/(V_{\mathcal{O}}/t^{r+1}V_{\mathcal{O}}, \psi)) \to \cdots .\end{aligned}$$

We will define the Hecke action in the next section.

2.4 Fourier transform

While the lattice model is defined without the assumption of V having a polarization, we want to show this construction is equivalent to the Schrödinger model in polarizable case.

By the colimit description of the category, it suffices to show $D(t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}})$ is equivalent to $D((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(V_{\mathcal{O}}/t^rV_{\mathcal{O}}, \psi))$. By taking Fourier transform, we know the latter is equivalent to $D((t^{-r}V_{\mathcal{O}}/t^rV_{\mathcal{O}})/(t^{-r}L_{\mathcal{O}}/t^rL_{\mathcal{O}}, \psi))$. Hence it suffices to show the following statement:

Proposition 1. *If choose a particular splitting of the short exact sequence* $0 \to L \to V \to V/L \to 0$, *we get a non-canonical equivalence of categories*

$$D(V/(L,\psi)) \cong D(V/L).$$

If the splitting preserves G*-action, we have* $D_G(V/(L,\psi)) \cong D_G(V/L)$.

Proof. Consider the space $L \times V/L$. It carries an L-action by $L \times L \times V/L \to L \times V/L$ by $(l_1, l_2, v+L) \mapsto (l_1 + l_2, v + L)$. From the map $L \times L \times V/L \to \mathbb{A}^1, (l_1, l_2, v + L) \mapsto \omega(l_1, v)$, we can define (L, ψ)-equivariant sheaves on $L \times V/L$.

Then we have the canonical equivalence $D(V/L) \simeq D((L \times V/L)/L) \simeq D((L \times V/L)/(L, \psi))$, where the second is given by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}_\psi$. This comes from $\mathcal{L}_\psi$ is (L, ψ)-equivariant, as $\omega(l_1 + l_2, v) = \omega(l_1, v) + \omega(l_2, v)$.

For a given section $V/L \to V$, we get a non-canonical isomorphism $V \cong L \times V/L$. This isomorphism makes the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{A}^1 & \longleftarrow & L \times V & \overset{\text{act}}{\underset{\text{proj}}{\rightrightarrows}} & V \\
\| & & \downarrow \cong & & \downarrow \cong \\
\mathbb{A}^1 & \longleftarrow & L \times L \times V/L & \overset{\text{act}}{\underset{\text{proj}}{\rightrightarrows}} & L \times V/L
\end{array}$$

This gives the equivalence $D(V/(L, \psi)) \cong D((L \times V/L)/(L, \psi))$.

If the G-action preserves the isomorphism $V \cong L \times V/L$, the above equivalences preserves G-actions. □

3 irreducible objects

3.1 singular support

Here we compute $T^*(V/(L, \psi))$. The character ω induces a map $\mathbb{A}^1 \times V \to \mathrm{Lie}(L)^* \simeq L^*$ given by $V \xrightarrow{\omega} V^* \to L^*$. The moment map of L-action $T^*V \to L^*$ is given by $(v, v^*) \mapsto (l \mapsto \langle l, v^* \rangle)$. Its fiber at $1 \in \mathbb{A}^1$ is

$$T^*V \times_{L^* \times V} (1 \times V) = \{(v, v^*) : \omega(v)|_L = v^*|_L\} = \{(v, v^*) : v - \omega^{-1}(v^*) \in L\}.$$

Here the last equation uses the fact that L is a Lagrangian, i.e.,

$$0 \to L \to V \simeq V^* \to L^* \to 0$$

is an exact sequence. Hence we have

$$T^*(V/(L, \psi)) = (T^*V \times_{L^* \times V} (1 \times V))/L \simeq V.$$

Similarly, we should expect $T^*(V_F/(V_\mathcal{O}, \psi)) \simeq V_F$. In fact, we see the singular support of sheaves in $D(V_F/(V_\mathcal{O}, \psi))$ lies in the colimit of the sets

$$\cdots \to \{L \subset t^{-r}V_\mathcal{O}/t^r V_\mathcal{O} \text{ is Lagrangian}\} \xrightarrow{p^* i_*} \{L \subset t^{-r-1}V_\mathcal{O}/t^{r+1}V_\mathcal{O} \text{ is Lagrangian}\} \to \cdots,$$

which is Lagrangians in V_F that contains some $t^N V_\mathcal{O}$.

Then we consider the behavior of $G_\mathcal{O}$-action on sheaves to its singular support.

Proposition 2. *The moment map of the* $G_\mathcal{O}$*-action is given by* $V_F \to \mathfrak{g}_\mathcal{O}^*, v \mapsto (g \mapsto \omega(v, gv))$.

Proof. First, for the finite case, if a group G acts on the symplectic space (V, ω) and fixes the Lagrangian L, we show the moment map of G-action on $V/(L, \psi)$ is by $V \to \mathfrak{g}^*, v \mapsto (g \mapsto \omega(v, gv))$.

The moment map of G-action on V is by $T^*V \to \mathfrak{g}, (v, v^*) \mapsto (gv, v^*)$. It restricts to a map from $T^*V \times_{L^* \times V} (1 \times V)$. The isomorphism $T^*V \times_{L^* \times V} (1 \times V) \simeq V$ is given by $(v, v^*) \mapsto \frac{1}{2}(v + \omega^{-1}(v^*))$ or $v \mapsto \{(v + l, \omega(v - l))\}/L$. Hence the image of V under the moment map is $g \mapsto \omega(g(v + l), v - l) = \omega(gv, v)$.

Then, for the local case, we have moment maps $t^{-r}V_{\mathcal{O}}/t^r V_{\mathcal{O}} \to \mathfrak{g}^*_{2r}, v \mapsto (g \mapsto \omega(v, gv))$. It is clear they are compatible for different r. By taking colimit, we get the desired moment map $V_F \to \mathfrak{g}^*_{\mathcal{O}}$. $\square$

3.2 relevant orbits

If a $(V_{\mathcal{O}}, \psi)$-equivariant sheaf on V_F is $G_{\mathcal{O}}$-equivariant, its singular support must be contained in the preimage of $0 \in \mathfrak{g}^*_{\mathcal{O}}$.

Any section $V_F/V_{\mathcal{O}} \to V_F$ induces a non-canonical equivalence $D(V_F/(V_{\mathcal{O}}, \psi))$ with $D(V_F/V_{\mathcal{O}})$, which does not preserve $G_{\mathcal{O}}$-action. However, by singular support calculation, we can still determine when a $G_{\mathcal{O}}$-orbit on $V_F/V_{\mathcal{O}}$ that could occur as the support of an irreducible object in $D_{G_{\mathcal{O}}}(V_F/(V_{\mathcal{O}}, \psi))$.

Proposition 3. *Let $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ and $n \leq m$. Consider the subset*

$$\{(v, v^*) : v^*v \in \mathfrak{gl}_n(\mathcal{O}), vv^* \in \mathfrak{gl}_m(\mathcal{O})\} \subset V(F) \times V^*(F)$$

and its image in $V(F)/V(\mathcal{O}) \times V^*(F)/V^*(\mathcal{O})$. Under suitable $\mathrm{GL}_n(\mathcal{O}) \times \mathrm{GL}_m(\mathcal{O})$-action, any element in the quotient can be conjugate to

$$\left(\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(t^{-b_1}, \dots, t^{-b_s}) \end{pmatrix} \right) \tag{1}$$

for $r + s \leq n, a_1 \geq \cdots \geq a_r \geq 1, b_s \geq \cdots \geq b_1 \geq 1$.

Proof. By row and column operators on an elements in $V(F)$, one can make it diagonal, i.e, of the form

$$\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_n}) \\ 0 \end{pmatrix}$$

for $a_1 \geq \cdots \geq a_n$. Let $r = \max\{i : a_r > 0\}$.

Write $v^* = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$. The condition $v^*v \in \mathfrak{gl}_m(\mathcal{O})$ and $vv^* \in \mathfrak{gl}_n(\mathcal{O})$ is equivalent to $x_{ij} \in t^{\max\{a_i, a_j\}}\mathcal{O}$. Hence v^* is of the form

$$\begin{pmatrix} A_{r,r} & A_{r,m-r} \\ A_{n-r,r} & A_{n-r,m-r} \end{pmatrix}$$

where $A_{i,j} \in \mathrm{Mat}_{i,j}(F)$ and $A_{r,r}, A_{r,m-r}, A_{n-r,r}$ has coefficients in $t\mathcal{O}$.

Next, use $\mathrm{GL}_{n-r}(\mathcal{O}) \times \mathrm{GL}_{m-r}(\mathcal{O})$ to do row and column operators to make $A_{n-r,m-r}$ diagonal. Thus we get $v^* + V^*(\mathcal{O})$ is conjugate to $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(t^{-b_1}, \dots, t^{-b_s}) \end{pmatrix} + V^*(\mathcal{O})$.

Since $v \in \begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$ and matrices $\begin{pmatrix} 1 & \\ & \mathrm{GL}_{n-r} \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ & \mathrm{GL}_{m-r} \end{pmatrix}$ fix this set, we know $v + V(\mathcal{O})$ is conjugate to $\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$. $\square$

Corollary 1. *Let $n \leq m$. The irreducible elements in $D_{\mathrm{GL}_{n\mathcal{O}} \times \mathrm{GL}_{m\mathcal{O}}}((T^*V)_F/(T^*V)_{\mathcal{O}})$ is indexed by $X_{\bullet}(\mathrm{GL}_n)$.*

Proof. Just note that the element in (1) corresponds to $(a_1, \dots, a_r, 0 \dots, 0, -b_1, \dots, -b_s)$ in $X_\bullet(\mathrm{GL}_n)$. □

Proposition 4. *Let $V = \mathrm{Hom}(\mathbb{C}^{2n}, \mathbb{C}^{2m})$ and $n \le m$. $\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ is equipped with standard symmetric inner product and $\mathbb{C}^{2m} = \mathbb{C}^m \oplus (\mathbb{C}^m)^*$ is equipped with standard anti-symmetric inner product. Consider the subset*

$$\{v \in V(F) : v^*v \in \mathfrak{so}_{2n}(\mathcal{O}), vv^* \in \mathfrak{sp}_{2m}(\mathcal{O})\}$$

and its image in $V(F)/V(\mathcal{O})$. Under suitable $\mathrm{O}_{2n}(\mathcal{O}) \times \mathrm{Sp}_{2m}$-action, any element in the quotient can be conjugate to

$$\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{diag}(t^{-b_1}, \dots, t^{-b_s}) \end{pmatrix}$$

for $r + s \le n, a_1 \ge \dots \ge a_r \ge 1$, $b_s \ge \dots \ge b_1 \ge 1$.

Proof. Write

$$v = (v_1, v_2, v_3, v_4) \in \mathrm{Hom}(F^n, F^m) \oplus \mathrm{Hom}(F^n, (F^m)^*) \oplus \mathrm{Hom}((F^n)^*, F^m) \oplus \mathrm{Hom}((F^n)^*, (F^m)^*),$$

and

$$v^* = (-v_4^{\mathrm{t}}, -v_2^{\mathrm{t}}, v_3^{\mathrm{t}}, v_1^{\mathrm{t}}) \in \mathrm{Hom}(F^m, F^n) \oplus \mathrm{Hom}(F^m, (F^n)^*) \oplus \mathrm{Hom}((F^m)^*, F^n) \oplus \mathrm{Hom}((F^m)^*, (F^n)^*).$$

Then the condition of $vv^* \in \mathfrak{sp}_{2m}(\mathcal{O})$ is equivalent to $v_1v_4^{\mathrm{t}} + v_3v_2^{\mathrm{t}}, v_1v_3^{\mathrm{t}} + v_3v_1^{\mathrm{t}}, v_2v_4^{\mathrm{t}} + v_4v_2^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O})$. The condition of $v^*v \in \mathfrak{so}_{2n}(\mathcal{O})$ is equivalent to $v_3^{\mathrm{t}}v_2 - v_4^{\mathrm{t}}v_1, v_3^{\mathrm{t}}v_4 - v_4^{\mathrm{t}}v_3, v_1^{\mathrm{t}}v_2 - v_2^{\mathrm{t}}v_1 \in \mathfrak{gl}_n(\mathcal{O})$.

Use elements in $\mathrm{GL}_n(\mathcal{O}), \mathrm{GL}_m(\mathcal{O})$ and permutations $(\mathbb{Z}/2\mathbb{Z})^n \ltimes \mathfrak{S}_n, (\mathbb{Z}/2\mathbb{Z})^m \ltimes \mathfrak{S}_m$, we can make v_1 diagonal and $v_t((v_1)_{jj}) \le v_t((v_2)_{ij})$, $v_t((v_1)_{ii}) \le v_t((v_3)_{ij})$.

In particular, write $v_1 = \begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_n}) \\ 0 \end{pmatrix}$ for $a_1 \ge \dots \ge a_n$. Let $r = \max\{i : a_r > 0\}$. Write v_2 as follows

$$\begin{pmatrix} t^{-a_1}x_{11} & \cdots & t^{-a_n}x_{1n} \\ \vdots & & \vdots \\ t^{-a_1}x_{m1} & \cdots & t^{-a_n}x_{mn} \end{pmatrix},$$

where $x_{ij} \in \mathcal{O}$. Then the condition $v_1^{\mathrm{t}}v_2 - v_2^{\mathrm{t}}v_1 \in \mathfrak{gl}_n(\mathcal{O})$ gives $x_{ij} - x_{ji} \in t^{a_i+a_j}\mathcal{O}, 1 \le i, j \le n$.

Take $y_{ij} = y_{ji} = x_{ji}$ for $i \le r, i \le j$ and $y_{ij} = 0$ for $i, j > r$. This gives an element Y in $\mathrm{Sym}^2 \mathcal{O}^m \subset \mathrm{Sp}_{2m}(\mathcal{O})$. Take the action, we get $x'_{ij} = x_{ij} - x_{ji}$ and $x'_{ji} = 0$ for $i \le r, i \le j$. Thus $(v'_2)_{ij} = t^{-a_j}(x_{ij} - x_{ji}) \in t^{a_i}\mathcal{O} \subset \mathcal{O}$ and $(v'_2)_{ji} = 0$ for $i \le r, i \le j$. When $i, j > r$, we have $(v'_2)_{ij} = (v_2)_{ij} = t^{-a_j}x_{ij} \in t^{-a_j}\mathcal{O} \subset \mathcal{O}$. In conclusion, we have $v'_2 \in \mathrm{Hom}(\mathcal{O}^n, (\mathcal{O}^m)^*)$.

Similarly, write

$$v_3 = \begin{pmatrix} t^{-a_1}x_{11} & \cdots & t^{-a_1}x_{1m} \\ \vdots & & \vdots \\ t^{-a_n}x_{n1} & \cdots & t^{-a_n}x_{nn} \\ x_{n+1,1} & \cdots & x_{n+1,n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix},$$

where $x_{ij} \in \mathcal{O}$ for $1 \le i, j \le n$. The condition $v_1v_3^{\mathrm{t}} + v_3v_1^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O})$ gives $x_{ij} + x_{ji} \in t^{a_i+a_j}\mathcal{O}$ for $1 \le i, j \le n$ and $x_{ij} \in t^{a_j}\mathcal{O}$ for $i > n$.

If $a_n \le 0$, from our construction of v_1, we know $x_{ij} \in \mathcal{O}$ for $i > n$. Otherwise, we have $a_1 \ge \cdots \ge a_n \ge 1$, then $x_{ij} \in t^{a_j}\mathcal{O} \subset \mathcal{O}$ for $i > n$. Anyway, we have $x_{ij} \in \mathcal{O}$ for $i > n$.

For the remaining, use exactly the same method as before to use an element in $\Lambda^2\mathcal{O}^n \subset \mathrm{SO}_{2n}(\mathcal{O})$ to make $v_3 \in \mathrm{Hom}((\mathcal{O}^n)^*, \mathcal{O}^m)$.

Now $v_3 v_2^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O}), v_3^{\mathrm{t}} v_2 \in \mathfrak{gl}_n(\mathcal{O})$, we get $v_1 v_4^{\mathrm{t}} \in \mathfrak{gl}_m(\mathcal{O}), v_4^{\mathrm{t}} v_1 \in \mathfrak{gl}_n(\mathcal{O})$. Use the result in Proposition 3, we can make v_4 into a diagonal matrix. □

Corollary 2. *Let $n \le m$. The irreducible elements in $D_{\mathrm{O}_{2n}\times \mathrm{Sp}_{2m}}(V_F/(V_{\mathcal{O}}, \psi))$ is indexed by $X_\bullet(\mathrm{O}_{2n})$.*

Proof. As $r + s \le n$, we can further use permutations in Weyl group to make $v + V(\mathcal{O})$ is conjugate to $\begin{pmatrix} \mathrm{diag}(t^{-a_1}, \dots, t^{-a_r}) & 0 \\ 0 & 0 \end{pmatrix} + V(\mathcal{O})$ for $r \le n$. Thus it corresponds to $(a_1, \dots, a_r, 0 \dots, 0) \in X_\bullet(\mathrm{O}_{2n})$. □

4 Hecke actions on lattice model

For a group homomorphism $G \to \widetilde{\mathrm{Sp}}(V)$, we want to define the action of $D(G)$ on $D(V/(L, \psi))$, we need a kernel sheaf on $G \times V$. This is done in [15] and also [19]. Let $\widetilde{\mathcal{L}}(V)$ be the space of all Lagrangians on V, [19] constructed a sheaf $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ on $\widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times V$ with properties. By the map $G \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V)$ given by $g \mapsto (gL, L)$, we obtain a sheaf $\mathcal{F}_G$ on $G \times V$. Thus we can define the action by

$$\mathcal{S} * \mathcal{F} = \mathrm{act}_!(\mathrm{pr}_2^* \mathcal{F}_G \otimes \mathrm{pr}_{23}^* \mathcal{S} \otimes \mathrm{pr}_{13}^* \mathcal{F} \otimes \mathcal{L}_\psi),$$

Here $\mathrm{act}\colon G \times V \times V \to V$ is given by $(g, v_1, v_2) \mapsto gv_1 + v_2$; pr are corresponding projections; $\mathcal{L}_\psi$ is the sheaf on $G \times V \times V$ given by the pullback of Artin-Schreier sheaf through $G \times V \times V \to \mathbb{A}^1, (g, v_1, v_2) \mapsto \omega(gv_1, v_2)$.

The properties of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ ensures this action gives a genuine module structure.

For the unit, take $\mathcal{S} = \delta_1 \in D(G)$. From the property $\Delta^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)} = \mathcal{F}_\Delta$, we know $\mathcal{F}_G|_1 = \Lambda_L$ and thus the convolution product with an (L, ψ)-equivariant sheaf is just identity.

Proposition 5. *The associativity holds. I.e., we have $\mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) \simeq (\mathcal{S}_1 * \mathcal{S}_2) * \mathcal{F}$.*

Proof. For clarity, we use $(g_1, g_2 v_1 + v_2, v_3)$ to denote the map $G \times G \times V \times V \times V \to G \times V \times V$ given by $(g_1, g_2, v_1, v_2, v_3) \mapsto (g_1, g_2 v_1 + v_2, v_3)$ and similarly for other maps. Then we have

$$\begin{aligned} \mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) =& (g_1(g_2 v_1 + v_2) + v_3)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F}) \otimes \\ & \otimes (g_2, v_2)^* \mathcal{F}_G \otimes (g_1, v_3)^* \mathcal{F}_G \otimes \omega(g_2 v_1, v_2)^* \mathcal{L}_\psi \otimes \omega(g_1(g_2 v_1 + v_2), v_3)^* \mathcal{L}_\psi). \end{aligned}$$

From the convolution property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$, we have the following isomorphism in $\widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times V$:

$$\mathrm{add}_!(\mathrm{pr}_{15}^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)} \otimes \mathrm{pr}_{34}^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)} \otimes \mathcal{L}_\psi) \simeq \mathrm{pr}_2^* \mathcal{F}_{\widetilde{\mathcal{L}}(V)}.$$

Take the pullback by the map $G \times G \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V), (g_1, g_2) \mapsto (g_1 g_2 L, g_1 L, L)$, we see

$$\mathrm{add}_!((g_1, v_1)^* \mathcal{F}_G \otimes (g_2, g_1^{-1} v_2)^* \mathcal{F}_G \otimes \mathcal{L}_\psi) \simeq \mathrm{mult}^* \mathcal{F}_G.$$

Here, we used the fact that $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ is G-equivariant. By change of variables, we see

$$(g_1, g_2, v_1 + g_1 v_2)_!((g_1, v_1)^* \mathcal{F}_G \otimes (g_2, v_2)^* \mathcal{F}_G \otimes \omega(v_1, g_1 v_2)^* \mathcal{L}_\psi) \simeq \mathrm{mult}^* \mathcal{F}_G.$$

Hence we can simplify, by letting $u = g_1v_2 + v_3$,

$$\begin{aligned}\mathcal{S}_1 * (\mathcal{S}_2 * \mathcal{F}) =&(g_1g_2v_1 + g_1v_2 + v_3)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F})\otimes\\ &\otimes (g_2, v_2)^*\mathcal{F}_G \otimes (g_1, v_3)^*\mathcal{F}_G \otimes \omega(g_1g_2v_1, g_1v_2 + v_3)^*\mathcal{L}_\psi \otimes \omega(g_1v_2, v_3)^*\mathcal{L}_\psi)\\ =&(g_1g_2v_1 + u)_!((g_1, g_2, v_1)^*(\mathcal{S}_1 \boxtimes \mathcal{S}_2 \boxtimes \mathcal{F}) \otimes (g_1g_2, u)^*\mathcal{F}_G \otimes \omega(g_1g_2v_1, u)^*\mathcal{L}_\psi)\\ =&(gv_1 + u)_!((g, v_1)^*((\mathcal{S}_1 * \mathcal{S}_2) \boxtimes \mathcal{F}) \otimes (g, u)^*\mathcal{F}_G \otimes \omega(gv_1, u)^*\mathcal{L}_\psi).\end{aligned}$$

The right hand side is exactly $(\mathcal{S}_1 * \mathcal{S}_2) * \mathcal{F}$. □

The image of an (L, ψ)-equivariant sheaf is still an (L, ψ)-equivariant sheaf comes from the act_{lr}-equivariant property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$.

If a subgroup $H \subset G$ fixes (L, ψ), we get the map $G/H \to \widetilde{\mathcal{L}}(V) \times \widetilde{\mathcal{L}}(V)$, using it, we can define the action of $D(H\backslash G/H)$ on $D_H(V/(L, \psi))$ similarly:

$$\mathcal{S} * \mathcal{F} = \mathrm{act}_!(\mathrm{pr}_2^*\mathcal{F}_G \otimes \mathrm{pr}_3^*(\mathcal{S}\widetilde{\boxtimes}\mathcal{F}) \otimes \mathcal{L}_\psi),$$

Here $\mathrm{act}\colon H\backslash((G \times^H V) \times V) \to H\backslash V$ is given by $(g, v_1, v_2) \mapsto gv_1 + v_2$. Since H fixes L, $\mathcal{F}_G$ descends to a sheaf $\mathcal{F}_{G/H}$ on $G/H \times V$. The act_G-equivariant property of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ ensures $\mathcal{F}_{G/H}$ is H-equivariant under the action of $h \cdot (gH, v) = (hgH, hv)$. In conclusion, the action $\mathcal{S} * \mathcal{F}$ is well-defined. The proof of properties such as associativity is identical as above.

The compatibility of $\mathcal{F}_{\widetilde{\mathcal{L}}(V)}$ under taking a subquotient $W^\perp/W$ of a Lagrangian $W \subset V$ ensures the actions of $\mathrm{Sat}_{G_{2r}}$ on $D_{G_{2r}}((t^{-r}V_\mathcal{O}/t^rV_\mathcal{O})/(V_\mathcal{O}/t^rV_\mathcal{O}, \psi))$ are compatible. In conclusion, we have the action of Sat_G on $D_{G_\mathcal{O}}(V_F/(V_\mathcal{O}, \psi))$.

For our case, $G \times H \to \mathrm{Sp}(V)$ has a lift to $\widetilde{\mathrm{Sp}}(V)$, we obtain a $D_{G_\mathcal{O}\times H_\mathcal{O}}(\mathrm{Gr}_{G\times H})$-action on $D_{G_\mathcal{O}\times H_\mathcal{O}}(V_F/(V_\mathcal{O}, \psi))$.

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